DERIVED SOLUTIONS OF DIFFERENTIAL EQUATIONS.

(Short Methods).

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When $F(m)=0$ has n roots equal a, then $F(m)=0$ and its first $n-1$ derivatives have this root in common. So if $F(m, x)$ is a solution of a differential equation, for different values of m, then, if n values of m are equal to a, are also the first $n-1$ partial derivatives of $F(m, x)$ with respect to m also, generally, solutions, when, after differentiation, m is changed to a.

CASE I.

(a) Linear differential equations with constant coefficients, second member zero.

$$y=e^{mx}$$

is the solution.

If $(D-a)^n$ is a factor of the first member, using the symbolic method of solution, then

$$e^{mx},\ xe^{mx},\ x^2e^{mx}\ \ldots\ldots\ldots,\ x^{n-1}e^{mx},$$

are all solutions, when m is changed to a, these being the partial derivatives with respect to m. Multiply each of these by an arbitrary constant and add for the general solution corresponding to $(D-a)^n$.

(b) When D^2+a^2 is a factor of the first member,

$$y=\sin\ mx,\quad y=\cos\ mx$$

are solutions, when m is a.

When $(D^2+a^2)^n$ is a factor of the first member,

$\sin\ ax$, $x\ \sin\ (\pi/2+ax)$, $x^2\sin\ (2\pi/2+ax)$, ...
$x^{n-1}\sin\ (\ (n-1)\ \pi/2+ax)$,
$\cos\ ax$, $x\ \cos\ (\pi/2+ax$, $x^2\cos\ (2\pi/2+ax)$, ...
$x^{n-1}\cos\ (\ (n-1)\ \pi/2+ax)$

are solutions.

(c) When $(D-a)^2+b^2$ is a factor of the first member,

$$y=e^{mx}$$

is a solution, when m is changed to $a+bi$ and $a-bi$, giving,

$$y=e^{ax}\cos bx\quad\text{and}\quad y=e^{ax}\sin\ bx$$

as solutions, after addition and subtraction.

When $((D-a)^2+b^2)^n$ is a factor of the first member, the $(n-1)$ partial derivatives of e^{mx} with respect to m,

$$xe^{mx},\ x^2e^{mx},\ \ldots\ldots\ldots,\ x^{n-1}e^{mx},$$

are also solutions, when m is $a+bi$, or $a-bi$, giving also as solutions,

$$xe^{ax}\sin bx,\ xe^{ax}\cos bx,\ x^2e^{ax}\sin bx,$$
$$x^2e^{ax}\cos bx,\ \ldots\ldots\ldots,\ x^{n-1}e^{ax}\sin bx,$$
$$x^{n-1}e^{ax}\cos bx.$$

Multiply each by an arbitrary constant and add.

Case II.

For the homogeneous linear differential equation of the form,

$$x^n\frac{d^ny}{dx^n}+x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+x\frac{dy}{dx}+y=0.$$

$$y=x^m$$

gives $x^n F(m)=0$, and if $m-a$ is a factor of $F(m)$, a solution is

$$y=x^a$$

The partial derviatives of x^m, with respect to m, are

$$x^m\log_e x;\ x^m\log^2_e x;\ x^m\log^3_e x,\ \text{etc.}$$

Thus if $(m-a)^n$ is a factor of $F(m)$, the solutions are,

$y=x^a$, $y=x^a\log_e x$, $y=x^a(\log_e x)^2$, ... $y=x^a(\log_e x)^{n-1}$

Multiply each by an arbitrary constant and add.

Case III.

(a) In (a) of Case I, if the second member is e^{mx} (instead of zero) *the particular solution* is

$$y=\frac{e^{mx}}{D-a}=\frac{e^{mx}}{m-a}$$

for the equation $(D-a)\,y=e^{mx}$.

This fails when $m=a$.

All failing cases of this sort will give a solution when treated like the form $\frac{0}{0}$ in calculus (differentiating as many times as the factor occurs, *omitting differentiation as to factors giving no trouble*, just as in the calculus problem).

(1)
$$y = \frac{e^{mx}}{m-a} = \frac{\frac{\partial}{\partial m}(e^{mx})}{\frac{\partial}{\partial m}(m-a)} = xe^{mx} = xe^{ax}$$

(2)
$$y = \frac{e^{mx}}{(m-a)^2} = \frac{xe^{mx}}{2(m-a)} = \frac{x^2e^{mx}}{2} = \frac{x^2e^{ax}}{2}$$

(3)
$$y = \frac{e^{mx}}{(m-a)^n} = \frac{x^n e^{ax}}{\underline{|n}}$$

(4)
$$y = \frac{e^{mx}}{(m^2+5m+3)(m-2)^2}, \text{ when } m = 2$$
$$= \frac{xe^{mx}}{(17)\,2\,(m-2)} = \frac{x^2e^{mx}}{(17)\,(2)} = \frac{x^2e^{2x}}{34}$$

(b) For $(D^2+a^2)\ y = \sin\,(mx)$

(1)
$$y = \frac{\sin\,(mx)}{D^2+a^2} = \frac{\sin\,(mx)}{-m^2+a^2}$$

This fails when $m=a$, but

(1)
$$y = \frac{\frac{\partial}{\partial m}(\sin\,(mx))}{\frac{\partial}{\partial m}(-m^2+a^2)} = \frac{x\cos\,(mx)}{-2m} = \frac{x\cdot\cos\,(ax)}{-2a}$$

(2)
$$y = \frac{\cos\,(mx)}{D^2+a^2} = \frac{\cos\,(mx)}{a^2-m^2} = \frac{x\sin\,(mx)}{2m} = \frac{x\sin\,(ax)}{2a},$$

when $m=a$.

(3)
$$y = \frac{\sin mx}{(D^2+9)\,(D^2+4)} = \frac{\sin mx}{(9-m^2)\,(4-m^2)}$$

For $m=2$,
$$y = \frac{x\cos mx}{5(-2m)} = \frac{-x\cos 2x}{20}$$

For $m=3$,
$$y = \frac{x\cos mx}{(-2m)\,(-5)} = \frac{x\cos 3x}{30}$$

(4)
$$y = \frac{\sin mx}{(D^2+4)^2}, \text{ when } m = 2$$
$$= \frac{\sin mx}{(4-m^2)^2} = \frac{x\cos mx}{(-4m)\,(4-m^2)} = \frac{-x^2\sin mx}{(-4m)\,(-2m)}$$
$$= \frac{-x^2\sin 2x}{32}$$

(5) $$y = \frac{\sin mx}{(D^2+4)^3}, \text{ when } m=2$$

$$= \frac{\sin mx}{(4-m^2)^3} = \frac{x \cos mx}{3(-2m)\ (4-m^2)^2} = \frac{-x^2 \sin mx}{\underline{|3}\ (-2m)^2(4-m^2)}$$

$$= \frac{-x^3 \cos mx}{\underline{|3}\ (-2m)^3} = \frac{x^3 \cos 2x}{\underline{|3}\ (4)^3}$$

(6) $$y = \frac{\sin mx}{(D^2+a^2)^n}, \text{ when } m=a$$

$$= \frac{x^n \sin\left(n\frac{\pi}{2}+mx\right)}{\underline{|n}\ 2^n\ (-m)^n} = \frac{x^n \sin\left(n\frac{\pi}{2}+ax\right)}{\underline{|n}\ 2^n\ (-a)^n}$$

Similar treatment when the second member is the cosine.

Case IV.

Linear differential equations, constant coefficients, second member a constant.

This is a particular case of (a) Case III.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 4 = 4e^{ox}$$

$$\therefore\ y = \frac{4e^{ox}}{D^2-5D+6} = \frac{4}{6} = \frac{2}{3},$$

when $D=0$, is the particular solution.

Case V.

Failing case of Case IV.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} = 4 = 4e^{ox}$$

$$y = \frac{4e^{mx}}{m^2-5m}$$

is the solution when $m=0$.

$$\therefore \text{ As in Case III, } y = \frac{4xe^{mx}}{2m-5}, \text{ when } m=0$$

$$\text{or, } y = -\frac{4x}{5},$$

is the *particular solution.*

Case VI.

The non homogeneous linear differential equation.

The solution is usually two or more series gotten from.

$y=x^m\ (A_0+A_1,\ x^{n_1}+A_2\ x^{n_2}+A_3x^{n_3}+$ etc.), (S), (where A_1, A_2, A_3, etc., are functions of m, and each of all the preceding by giving m particular values gotten by substituting x^m for y in the given equation.

Calling (S), $y=F\ (m, x)$, then, in case of two equal roots in m, is

$$y = \frac{\partial}{\partial m} F\ (m, x).$$

a solution, if, after differentiation, m is given the value of this root. And in case of three equal roots, is also

$$y = \frac{\partial^2}{\partial m^2} F\ (m, x)$$

a solution, and so on.

Suppose k is a particular value of m, and that x^m, A_1, A_2, etc., are all expressed in powers of $m-k$:

$$x^m=x^k\,(1+(m-k)\log_e x+\frac{(m-k)^2}{2}\log^2{}_e x+\text{etc.})$$

$$A_0=A_0(1+\text{Zero}\ (m-k)+\text{Zero}\ (m-k)^2+\text{ etc}).$$
$$A_1=A_0(a_1+b_1\ (m-k)+c_1\ (m-k)^2+\text{ etc.})$$
$$A_2=A_0(a_2+b_2\ (m-k)+c_2\ (m-k)^2+\text{etc.}),$$

and so on.

Substitute these values for the A's in S and calling $1+a_1x^{n_1}+a_2x^{n_2}+$ etc., the A-series; the coefficient of $m-k$, the B-series; that of $(m-k)^2$, the C-series, etc., (S) becomes:

$$\begin{aligned} y=&A_0x^k\ (1+\ (m-k)\ \log_e x+\text{etc.})\text{ times}\\ &(\text{A-series}+\ (m-k)\ (\text{B-series})\ +\ (m-k)^2\ (\text{C-series})\\ &+\text{ etc.})\ =\ A_0x^k\ (\text{A-series})\\ &+A_0x^k\ (\ (\text{A-series})\ \log_e x\ +\ \text{B-series})\ (m-k)\\ &+\ A_0\ x^k\ (\frac{(\text{A-series})}{2}\log^2{}_e x+\ (\text{B-series})\ \log_e x\\ &+\ \text{C-series})\ (m-k)^2+\text{etc.} \end{aligned}$$

In case m has the value k only once,

$$y=A_0\ x^k\ \text{A-series} \qquad (1)$$

is the solution.

If m has the value of k twice,

$$y=B_0x^k\ (\ (\text{A-series})\ \log_e x+\text{B-series}) \qquad (2)$$

is also a solution. This is

$$y = \frac{\partial}{\partial m} F\ (m, x)$$

when $m=k$.

And thus is

$$y = x^k\ (A_o + B_o \log_e x)\ \text{A-series} + B_o x^k\ (\text{B-series})$$

a solution.

And in case m has the value k three times, then in addition to the solutions (1), (2), is also

$$y = C_o x^k \frac{(\text{A-series})}{2} \log^2{}_e x + (\text{B-series}) \log_e x + (\text{C-series})$$

a solution. This is

$$y = \frac{\partial^2}{\partial m^2} F\ (m, x) \text{ when } m = k$$

And thus is

$$y = x^k\ (A_o + B_o \log_e x + \frac{C_o}{2} \log^2{}_e x)\ (\text{A-series})$$
$$+ x^k (B_o + C_o \log_e x)\ (\text{B-series})$$
$$+ x^k\ (\text{C-series})$$

a solution. And so on.

It is quite easy, in any particular case, to obtain the B-series, C-series, etc., from the A-series, far easier than by the method usual in the texts (compare the following with Johnson, p. 181 to 194).

Illustrations.

(a) Series starting with x^o.

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

$$y = x^m, \text{ gives } m^2 x^{m-1} + x^m = 0 \qquad (1)$$

We may assume an ascending series, starting with x^o, consecutive powers differing by unity.

If we assume $y = \sum_{0}^{\infty} A_r x^{m+r}$,

then by (1)

$$(m+r)^2\ A_r + A_{r-1} = 0$$

$$\therefore\ A_r = -A_{r-1} \frac{1}{(m+r)^2}\ (2)$$

For $m = 0$, $A_r = -A_{r-1} \cdot \frac{1}{r^2}$ (3);

(3) gives the A-series,

$$y = 1 - \frac{x}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^2.} + \text{etc.}$$

If we differentiate the m-factor, $-\frac{1}{(m+r)^2}$, occurring in (2), we have

$$\left(\frac{1}{(m+r)^2}\right)\cdot\left(\frac{2}{m+r}\right)$$

And for $m=0$, this is

$$\left(-\frac{1}{r^2}\right)\left(-\frac{2}{r}\right)$$

And the n^{th} term of the B-series comes at once from the $(n+1)^{th}$ term of the A-series (beginning with the 2d term of the A-series) by multiplying by

$$\sum_1^n\left(-\frac{2}{r}\right)$$

So that the B-series is

$$2\frac{x}{1^2}-3\frac{x^2}{1^2.2^2}+\frac{11}{3}\cdot\frac{x^3}{1^2.2^2.3^2}-\frac{25}{6},\frac{x^4}{1^2.2^2.3^2.4^2}+\text{ etc.}$$

As this method of getting the B-series from the A-series gives the relation of corresponding terms, it makes the settlement of the question of the convergency of the B-series much easier than Johnson's method. Into the question of convergency I am not entering here, but merely showing how to get the B-series, whether or not it is a usable solution.

The reason for the above procedure is this: The A's come each from the preceding by multiplication:

$$\begin{aligned}A_1&=A_o(1+g_1(m-k)+\text{ etc.})\\ A_2&=A_1(1+g_2(m-k)+\text{etc.})\\ &=A_o(1+(g_1+g_2)\ (m-k)+\text{etc.})\\ A_3&=A_2(1+g_3(m-k)+\text{etc.})\\ &=A_o(1+(g_1+g_2+g_3)\ (m-k)+\text{etc.})\end{aligned}$$

and so on.

The coefficient of $m-k$ in A_n is $\sum_1^n g_n$.

And since, in the expansion, the coefficients of $m-k$ are values of $F'(m)$, the B-series comes from the A-series by the relation $\sum_1^n F'(m)$ using the m factor of A_{r-1}, as in the foregoing example.

$$(2)\quad x(1-x^2)\frac{d^2y}{dx^2}+(1-3x^2)\frac{dy}{dx}-xy=0\ \text{(Johnson-181)}$$

$$y=x^m\text{ gives }m^2x^{m-1}-(m+1)^2\,x^{m+1}=0\ \text{(E)}$$

And we may thus assume an ascending series beginning with $m=0$, or a descending series beginning with $m=-1$, exponents differing by 2 in each case.

(1) The ascending series,

$$y = \sum_{0}^{\infty} A_r x^{m+2r}$$

gives, by (E),

$$(m+2r)^2 A_r - (m+2r-1)^2 A_{r-1} = 0$$

$$\therefore A_r = \frac{(m+2r-1)^2}{(m+2r)^2} \cdot A_{r-1} \quad (2)$$

$$\text{For } m=0,\ A_r = \frac{(2r-1)^2}{(2r)^2} A_{r-1} \quad (3);$$

(3) gives the A-series.

Differentiating the m-factor of (2) we have

$$\left(\frac{(m+2r-1)^2}{(m+2r)^2}\right) \left(\frac{2}{(m+2r)\ (m+2r-1)}\right)$$

And for $m=0$, this is

$$\left(\frac{(2r-1)^2}{(2r)^2}\right) \left(\frac{1}{r(2r-1)}\right)$$

And thus the n^{th} term of the B-series is derivable from the $(n+1)^{th}$ term of the A-series, beginning with the 2d, by multiplying by

$$\sum_{r=1}^{r=n} \frac{1}{r(2r-1)} \quad (4)$$

By (3) the A-series is

$$y = 1 + \frac{1^2}{2^2} x^2 + \frac{1^2.3^2.}{2^2.4^2} x^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2.} x^6 + \text{etc.}$$

And by (4) the B-series is

$$\frac{1^2}{2} x^2 + \frac{7}{6} \frac{1^2.3^2.}{2^2.4^2.} x^4 + \frac{37}{30} \cdot \frac{1^2.3^2.5^2.}{2^2.4^2.6^2.} x^6 + \text{etc.}$$

(2) The descending series for the same equation.

$(m+1)^2 x^{m+1} - m^2 x^{m-1} = 0$, gives,

$(m-2r+1)^2 x^{m-2r} A_r - (m-2r+2) x^{m-2r-1} A_{r-1} = 0$

$$\therefore A_r = \frac{(m-2r+2)^2}{(m-2r+1)^2} A_{r-1} \quad (1)$$

$$\text{For } m=-1, \qquad A_r = \frac{(2r-1)^2}{(2r)} A_{r-1} \quad (2)$$

This gives the A-series.

Differentiating the m-factor of (1)

$$\left(\left(\frac{m-2r+2}{m-2r+1}\right)^2\right)\quad\left(\frac{-2}{(m-2r+1)\ (m-2r+2)}\right)$$

For $m = -1$, this is

$$\left(\left(\frac{2r-1}{2r}\right)^2\right)\quad\left(-\frac{2}{2r\ (2r-1)}\right)$$

And thus the n^{th} term of the B-series is derived from the $(n+1)^{th}$ term of the A-series, beginning with the 2d, by multiplying by

$$\sum_1^n \frac{-2}{(2r-1)\ (2r)}$$

The A-series is

$$x^{-1}\ (1+\frac{1^2}{2^2}x^{-2}+\frac{1^2.3^3.}{2^2.4^2}x^{-4}+\text{etc.})$$

And the B-series is

$$-2x^{-1}\ (\frac{1^2}{2^2}\cdot\frac{1}{1.2}x^{-2}+\frac{1^2.3^3}{2^2.4^2}\left(\frac{1}{1.2}+\frac{1}{3.4}\right)x^{-4}+\text{etc.})$$

(b) Series starting with the same value of m, but not zero.

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+(4-x)\ y=0$$

$y=x^m$ gives $(m-2)^2\ x^m-x^{m+1}=0$.

$\therefore$ Two ascending series, starting with x^2,

$$A_r=\frac{A_{r-1}}{(m+r-2)^2}\qquad(1)$$

$$\text{For } m=2,\ A_r=,\ A_{r-1}\left(\frac{1}{r^2}\right)\qquad(2);$$

(2) gives the A-series.

Differentiating the m-factor of (1),

$$\left(\frac{1}{(m+r-2)^2}\right)\quad\left(\frac{-2}{m+r-2}\right)$$

For $m=2$, this is

$$\left(\frac{1}{r^2}\right)\quad\left(-\frac{2}{r}\right)$$

The n^{th} term of the B-series is derivable from the $(n+1)^{th}$ term of the A-series by multiplying by

$$\sum_1^n\left(-\frac{2}{r}\right)$$

The A-series is

$$x^2\left(1+\frac{1}{1^2}x+\frac{1}{1^2.2^2}x^2+\frac{1}{1^2.2^2.3^3}x^3+\text{etc.}\right)$$

And the B-series is

$$-2x^2\left(\frac{x}{1^2}+\frac{3}{2}\cdot\frac{1}{1^2.2^2}x^2+\frac{11}{6}\cdot\frac{1}{1^2.2^2.3^2}x^3+\text{etc.}\right)$$

Note—Equations of the sort that give two or more solutions starting with a value of m different from zero, can be reduced to the latter sort in two ways. First, by a change of the dependent variable. If the starting value is x^k, set $y=x^k y$. The equation just considered, if we set $y=x^2y$, becomes

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0,$$

and $x^m=y$, gives $m^2\,x^{m-1}-x^m=0$

$$\therefore\ A_r=\frac{1}{(m+r)^2}\cdot A_{r-1}$$

$$\text{For } m=0,\ A_r=A_{r-1}\left(\frac{1}{r^2}\right)$$

This gives the A-series.

Differentiating the m-factor,

$$\left(\frac{1}{(m+r)^2}\right)\ \left(\frac{-2}{m+r}\right)$$

becomes, for $m=0$,

$$\left(\frac{1}{r^2}\right)\ \left(\frac{-2}{r}\right)$$

The B-series is derivable from the A-series by

$$\sum_{1}^{n}\left(\frac{-2}{r}\right)$$

The resulting two series multiplied by x^2, give the required A-series and B-series.

Or, we might change $m-k$ to n, and deduce the series in terms of n as heretofore in terms of m. Thus case (b) is always reducible to case (a).

(c) Series starting with values of m differing by a multiple of s when

$$y=\sum_{0}^{\infty}A_r x^{m+rs}$$

is the assumed solution.

(Compare Johnson, p. 185–9).

$$x^2\,(1+x)\,\frac{d^2y}{dx^2}+x\,\frac{dy}{dx}+(1-2x)\,y=0$$

(Compare Johnson, p. 185).

$y=x^m$, gives

$$(m+1)\,(m-2)\,x^{m+1}+(m^2+1)\,x^m=0 \qquad (1)$$

And we may select two descending series, beginning with x^{-1} and x^2, with powers differing by unity.

By (1), $y=\sum_{1}^{\infty} A_r x^{m-r}$ gives

$$(m-r)\,(m-r-3)\,A_{r+1}+(\,(m-r)^2+1)\,A_r=0$$

$$\therefore A_{r+1}=-\frac{(m-r)^2+1}{(m-r)\,(m-r-3)}A_r \qquad (2)$$

For $m=2$, this gives,

$$A_{r+1}=\frac{(2-r)^2+1}{(2-r)\,(r+1)}A_r$$

And this will fail when $r=2$

For $m=-1$, (2) gives

$$A_{r+1}=-\frac{(r+1)^2+1}{(r+1)\,(r+4)}A_r \qquad (3)$$

This gives the A-series, from which, by the method already used, we can also get the B-series.

Differentiating the m-factor of (2)

$$\left(-\frac{(m-r)^2+1}{(m-r)\,(m-r-3)}\right)\left(-\frac{3(m-r)^2+2(m-r)-3}{(m-r)\,(m-r-3)\,(\,(m-r)^2+1)}\right)$$

When $m=-1$, this becomes

$$\left(-\frac{(1+r)^2+1}{(r+1)\,(r+4)}\right)\left(-\frac{3r^2+4r-2}{(r+1)\,(r+4)\,(r^2+2r+2)}\right)$$

Thus the terms of the B-series (except as to those preceding the A-series) are gotten, the n^{th} term of the B-series from the $(n+1)^{th}$ term of the A-series by multiplying by

$$-\sum_{0}^{n}\frac{3r^2+4r-2}{(r+1)\,(r+4)\,(r^2+2r+2)}$$

The A-series is

$$x^{-1}(1-\frac{2}{1.4}x^{-1}+\frac{2}{1.4}\cdot\frac{5}{2.5}\cdot x^{-1}+\text{ etc.}$$

And the B-series is

$$x^{-1}\left(-\frac{1}{4}\cdot\frac{2}{1.4}x^{-1}+\frac{3}{20}\cdot\frac{2}{1.4}\cdot\frac{5}{2.5}x^{-2}-\text{ etc.}\right.$$

To get the terms of the B-series preceding the A-series,

$$A_r=-\frac{(m-r)\,(m-r-3)}{(m-r)^2+1}A_{r+1},\text{ by (2)}$$

Differentiating

$$A_r=-\frac{(m-r)\,(m-r-3)}{(m-r)^2+1)}A_{r+1}\cdot\left(\frac{3(m-r)^2+2(m-r)-3}{(m-r)\,(m-r-3)\,(\,(m-r)^2+1)}\right)$$

And when $m=\ -1$, this is

$$A_r=-\frac{(1+r)\,(4+r)}{r^2+2r+2}\cdot A_{r+1}\left(\frac{3r^2+4r-2}{(1+r)\,(4+r)\,(r^2+2r+2)}\right)$$

$$\therefore\ A_r=A_{r+1}\left\{-\frac{(1+r)\,(4+r)}{r^2+2r+2}-\frac{3r^2+4r-2}{(r^2+2r+2)^2}(m+1)+\text{ etc.}\right\}$$

$$\therefore\ A_{-1}=A_0\,(0+3\,(m+1)$$
$$A_{-2}=A_{-1}\,(1+k_1\,(m+1)+\text{ etc.}$$
$$=A_0\,(0+3\,(m+1)+\text{ etc.})$$
$$A_{-3}=A_{-2}\,(\frac{2}{5}+k_2\,(m+1)+\text{ etc.})$$
$$=A_0\,(0+\frac{6}{5}(m+1)+\text{ etc.})$$
$$A_{-4}=A_{-3}\,(0+k_3\,(m+1)+\text{etc.})$$
$$=A_0\,(0+0\,(m+1);$$

and all subsequent terms vanish as to $m+1$.

$\therefore$ terms to be added are

$$B_0\,(3+3x+\frac{6}{5}x^2)$$

Vanishing of the first term in A_{-1} prevents using the procedure used heretofore, since in this case a summation no longer holds.

(2)
$$4x\,(1-x)\frac{d^2y}{dx^2}-4\frac{dy}{dx}-y=0$$

$y=x^m$ gives

$$4m\,(m-2)\,x^{m-1}-(2m-1)^2\,x^m=0$$

$\therefore$ series start with x^0 and x^2; exponents of powers differ by unity.

$$A_r = \frac{(2(m+r)-3)^2}{4(m+r)\ (m+r-2)} A_{r-1} \qquad (1)$$

For $m=0$, the series would fail for $r=2$.
For $m=2$

$$A_r = \frac{(2r+1)^2}{2r \,.\, 2(r+1)} A_{r-1} \qquad (2)$$

This gives the A-series.
Differentiating (1)

$$A_r = \frac{(2(m+r)-3)^2}{4(m+r)\ (m+r-2)} A_{r-1} \frac{2(m-r)-6}{(m+r)\ (m+r-2)\ (2(m+r)-3)}$$

And when $m=2$, this is

$$A_r = \frac{(2r-1)^2}{2r \,.\, 2(r+2)} A_{r-1} \cdot \frac{2(r-1)}{r(r+2)\ (2r+1)}$$

The B-series derivable from the A-series is gotten thus: The n^{th} term of the B-series from the $(n+1)^{th}$ term of the A-series by multiplying by

$$\sum_1^n \frac{2(r-1)}{r(r+2)\ (2r+1)}$$

And the terms of the B-series preceding the A-series come from

$$A_{r-1} = A_r \left(\frac{2r \,.\, 2(r+2)}{(2r+1)} - \frac{8(r+1)}{(2r+1)3}\ (m-2) \right)$$

$$\begin{aligned} \therefore\ A_{-1} &= A_o\ \ (0+8(m-2)\ ) \\ A_{-2} &= A_{-1}(-4-16(m-2)\ ) \\ &= A_o(0-32(m-2)\ ) \\ A_{-3} &= A_{-2}(0+k(m-2) \\ &= A_o(0+0(m-2)\ ) \end{aligned}$$

$\therefore$ the terms are

$$B_o(-32+8x)$$